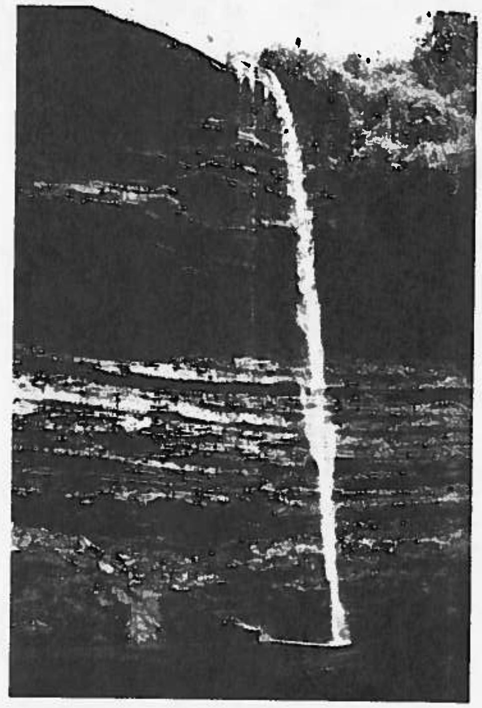
6.2 Solve Quadratic Equations



The flight path of a toy rocket, a ball, or any projectile can be predicted using a quadratic model. This model can also be used to determine when and where a projectile will land. For example, the path of a stone thrown into a ravine is modelled by the quadratic relation $y = -x^2 + 5x + 84$, where x is the distance, in metres, it travels horizontally and y is the height, in metres, above the river at the bottom of the ravine. You can solve for x using a variety of methods, including factoring a **quadratic equation** .

quadratic equation

- an equation in the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$

Investigate

How can factoring help you solve a quadratic equation?

1. If the product of two numbers is zero, what must be true about one or both of the numbers?
2. If $a \times b = 0$, what must be true about the value of a or the value of b or both?
3. How do steps 1 and 2 relate to solving the equation $(x - 3)(x + 5) = 0$?
4. **a)** Solve the equation $(x - 3)(x + 5) = 0$.
   b) Explain why your solutions cause the left side of the equation to equal zero.
5. Use your method to solve each equation.
   a) $(x + 2)(x + 9) = 0$ **b)** $(2x + 5)(3x - 4) = 0$
6. **a)** How is the equation $x^2 + 6x + 8 = 0$ different from those in steps 3 to 5? Describe the extra steps needed to solve the equation.
   b) Solve the equation.
7. Reflect Describe how you can use factoring to solve a quadratic equation.

Example 1 Solve by Factoring

Solve for x. Check your answers by substitution.

a) $x^2 + 9x + 14 = 0$

b) $2x^2 + 5x = 0$

c) $6x^2 - x = 15$

Connections

To solve an equation means to find the values of the variable that make the statement true. This is also called finding the roots of the equation.

Solution

a) $x^2 + 9x + 14 = 0$

$$(x + 7)(x + 2) = 0$$

Factor the left side.

$$x + 7 = 0 \quad \text{or} \quad x + 2 = 0$$
$$x = -7 \quad \text{or} \quad x = -2$$

One factor or the other must equal zero.

Check.

For $x = -7$:

$$\begin{aligned} \textbf{L.S.} &= x^2 + 9x + 14 \qquad \textbf{R.S.} = 0 \\ &= (-7)^2 + 9(-7) + 14 \\ &= 49 - 63 + 14 \\ &= 0 \end{aligned}$$

L.S. = R.S.

For $x = -2$:

$$\begin{aligned} \textbf{L.S.} &= x^2 + 9x + 14 \qquad \textbf{R.S.} = 0 \\ &= (-2)^2 + 9(-2) + 14 \\ &= 4 - 18 + 14 \\ &= 0 \end{aligned}$$

L.S. = R.S.

The solutions, or **roots**, are -7 and -2.

root (of an equation)

- the value of the variable that makes an equation true
- the same as the solution of an equation

b) $2x^2 + 5x = 0$

$$x(2x + 5) = 0$$

Remove the common factor.

$$x = 0 \quad \text{or} \quad 2x + 5 = 0$$
$$2x = -5$$
$$x = -\frac{5}{2}$$

One factor or the other must equal zero.

The roots are 0 and $-\frac{5}{2}$. The check of these roots is left to the reader.

c)

$$\begin{aligned} 6x^2 - x &= 15 \\ 6x^2 - x - 15 &= 0 \\ 6x^2 - 10x + 9x - 15 &= 0 \\ 2x(3x - 5) + 3(3x - 5) &= 0 \\ (3x - 5)(2x + 3) &= 0 \end{aligned}$$

Write in the form $ax^2 + bx + c = 0$. To factor the left side, break up the middle term and factor by grouping.

To factor $6x^2 - x - 15$, I must find two integers whose product is $6 \times (-15)$, or -90, and whose sum is -1. The integers -10 and 9 work.

$$3x - 5 = 0 \quad \text{or} \quad 2x + 3 = 0$$
$$3x = 5 \quad \text{or} \quad 2x = -3$$
$$x = \frac{5}{3} \quad \text{or} \quad x = -\frac{3}{2}$$

One factor or the other must equal zero.

Check.

For $x = \frac{5}{3}$:

$$\begin{aligned} \text{L.S.} &= 6x^2 - x \qquad \text{R.S.} = 15 \\ &= 6\left(\frac{5}{3}\right)^2 - \frac{5}{3} \\ &= 6\left(\frac{25}{9}\right) - \frac{5}{3} \\ &= \frac{50}{3} - \frac{5}{3} \\ &= 15 \end{aligned}$$

L.S. = R.S.

For $x = -\frac{3}{2}$:

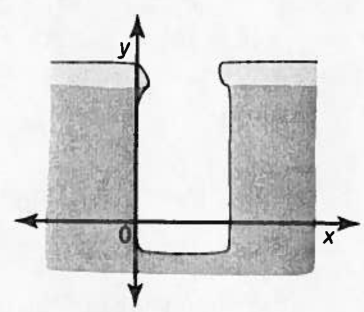
$$\begin{aligned} \text{L.S.} &= 6x^2 - x \qquad \text{R.S.} = 15 \\ &= 6\left(-\frac{3}{2}\right)^2 - \left(-\frac{3}{2}\right) \\ &= 6\left(\frac{9}{4}\right) + \frac{3}{2} \\ &= \frac{27}{2} + \frac{3}{2} \\ &= 15 \end{aligned}$$

L.S. = R.S.

The roots are $\frac{5}{3}$ and $-\frac{3}{2}$.

Example 2 Path of a Stone

The path of a stone thrown into a ravine is modelled by the quadratic relation $y = -x^2 + 5x + 84$, where x represents the distance, in metres, travelled horizontally and y represents the height, in metres, above the surface of the river at the bottom of the ravine. How far does the stone travel horizontally before it hits the water?



Solution

When the stone hits the water, its height is 0 m. So, let $y = 0$.

$$-x^2 + 5x + 84 = 0$$

$$\frac{-x^2}{-1} + \frac{5x}{-1} + \frac{84}{-1} = \frac{0}{-1} \qquad \text{Divide both sides of the equation by } -1.$$

$$x^2 - 5x - 84 = 0$$

$$(x + 7)(x - 12) = 0 \qquad \text{Factor the left side.}$$

$$x + 7 = 0 \quad \text{or} \quad x - 12 = 0 \qquad \text{One factor or the other must equal zero.}$$

$$x = -7 \quad \text{or} \quad x = 12$$

Since x represents a distance, it cannot be negative. So, reject the root -7.

Check $x = 12$.

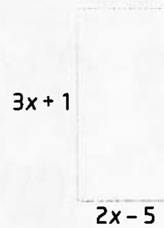
$$\begin{aligned} \text{L.S.} &= -x^2 + 5x + 84 \qquad \text{R.S.} = 0 \\ &= -(12)^2 + 5(12) + 84 \\ &= -144 + 60 + 84 \\ &= 0 \end{aligned}$$

L.S. = R.S.

The stone travelled 12 m horizontally before it hit the water.

Example 3 Dimensions of a Rectangle

A rectangle has dimensions $3x + 1$ and $2x - 5$. Its area is 1150 cm^2. What are its dimensions?



Solution

Method 1: Use Pencil and Paper

Substitute the area and expressions for the dimensions into the formula $A = l \times w$.

$$1150 = (3x + 1)(2x - 5)$$

$$1150 = 6x^2 - 13x - 5$$ Expand and simplify the right side.

$$6x^2 - 13x - 1155 = 0$$ Write in the form $ax^2 + bx + c = 0$.

$$6x^2 - 90x + 77x - 1155 = 0$$ Factor the left side.

$$6x(x - 15) + 77(x - 15) = 0$$

$$(x - 15)(6x + 77) = 0$$

$$x - 15 = 0 \quad \text{or} \quad 6x + 77 = 0$$ One factor or the other must equal zero.

$$x = 15 \quad \text{or} \quad x = -\frac{77}{6}$$

Since the dimensions cannot be negative, reject the solution $-\frac{77}{6}$.

Check $x = 15$.
Find the dimensions of the rectangle.

$3x + 1$	$2x - 5$
$= 3(15) + 1$	$= 2(15) - 5$
$= 46$	$= 25$

Check that these dimensions give an area of 1150 cm^2.
$46 \times 25 = 1150$
Therefore, the dimensions of the rectangle are 46 cm by 25 cm.

Method 2: Use a Computer Algebra System (CAS)

In the Home screen, use the **Expand** function with the area equation $(3x + 1)(2x - 5) = 1150$.